

ON NEW SUM-PRODUCT TYPE ESTIMATES

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ABSTRACT. New lower bounds involving sum, difference, product, and ratio sets for a set $A \subset \mathbb{C}$ are given. The estimates involving the sum set match, up to constants, the one obtained by Solymosi ([15]) for the reals and are obtained by generalising his approach to the complex plane. The bounds involving the difference set are slightly weaker. They improve on the best known ones, including the case $A \subset \mathbb{R}$, which also due to Solymosi ([14]), by means of combining the use of the Szemerédi-Trotter theorem with an arithmetic combinatorics technique.

1. INTRODUCTION

This manuscript subsumes the earlier homonymous preprint arXiv: math: 1111.4977 of the Second Author, which contained weaker estimates involving the sum-set.

Erdős and Szemerédi ([3]) conjectured that if A is a set of integers, then

$$|A + A| + |A \cdot A| \gg |A|^{2-o(1)},$$

where

$$A + A = \{a_1 + a_2 : a_{1,2} \in A\}$$

is called the sum set of A , the product $A \cdot A$, difference $A - A$, and ratio $A : A$ sets being similarly defined. (In the latter case one should not divide by zero.) The notations \ll , \gg , \approx are being used throughout to suppress absolute constants in inequalities, the symbol $o(1)$ in exponents absorbs logarithmic factors in the asymptotic parameter $|A|$, the cardinality of A .

Variations of the Erdős-Szemerédi question consider the set A living in other rings or fields, see e.g. [18], as well as replacing, e.g., the sum set with the difference set $A - A$. The conjecture is far from being settled, and therefore partial current “word records” on it vary with such variations of the input.

The best result for $A \subset \mathbb{R}$, for instance, is due to Solymosi ([15]), claiming

$$(1) \quad |A + A| + |A \cdot A| \gg |A|^{1+\frac{1}{3}-o(1)},$$

and would include the endpoint exponent $\frac{4}{3}$ if $A \cdot A$ were replaced by $A : A$.

However, the construction in [15] may appear to be specific for reals, and not to allow for replacing the sum set $A + A$ with the difference set $A - A$. So, if $A \subset \mathbb{C}$ or if $A + A$ for reals gets replaced by $A - A$, the best known result comes from an older paper of Solymosi ([14]), claiming

$$(2) \quad |A - A| + |A \cdot A| \gg |A|^{1+\frac{3}{11}-o(1)},$$

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also without the $o(1)$ term in the exponent if $A \cdot A$ gets replaced by $A : A$.

In this paper we show, firstly, that the main order-based consideration which enabled Solymosi to prove (1), namely the fact that if, for real positive a, b, c, d one has $\frac{a}{b} < \frac{c}{d}$, then the ratio $\frac{a+c}{b+d}$ falls in between, admits a natural extension to the complex case. We thereby generalise the estimate (1) to the case $A \subset \mathbb{C}$. Secondly, we prove new estimates involving the difference set, for $A \subset \mathbb{C}$, which improve on (2). For this we use rather different considerations, relying on the Szemerédi-Trotter theorem, plus an arithmetic technique.

Theorem 1. *For any $A \subset \mathbb{C}$, with two or more elements one has the following estimates:*

$$(3) \quad \begin{aligned} |A + A| + |A : A| &\gg |A|^{1+\frac{1}{3}}, \\ |A + A| + |A \cdot A| &\gg |A|^{1+\frac{1}{3}-o(1)}, \end{aligned}$$

as well as

$$(4) \quad \begin{aligned} |A - A| + |A : A| &\gg |A|^{1+\frac{9}{31}-o(1)}, \\ |A - A| + |A \cdot A| &\gg |A|^{1+\frac{11}{39}-o(1)}, \end{aligned}$$

Since the proofs of the former two and the latter two estimates in Theorem 1 use different techniques, the main body of the proof of Theorem 1 is presented in two separate and largely independent sections.

2. PROOF OF THEOREM 1

Since $|A| \geq 2$, and absolute constants are suppressed in the key estimates (3),(4), we can assume that $0 \notin A$. Consider the point set $A \times A$ in the coordinate plane \mathbb{C}^2 .

The multiplicative energy $E_*(A)$ of A is defined as the number of solutions of the following equation

$$E_*(A) = |\{(a_1, \dots, a_4) \in A \times \dots \times A : a_1/a_2 = a_3/a_4\}|.$$

By the Cauchy-Schwarz inequality

$$(5) \quad E_*(A) \geq \max \left(\frac{|A|^4}{|A \cdot A|}, \frac{|A|^4}{|A : A|} \right).$$

Geometrically, $E_*(A)$ is the number of ordered pairs of points of $A \times A$ in the plane on straight lines through the origin, whose slopes are members of the ratio set $A : A$. Hence, elements of $A : A$ can be identified with lines through the origin supporting points of $A \times A$.

As usual, the proof will single out and deal with a “popular” subset L of these lines (i.e. ratios), the notation P further standing for a subset of $A \times A$ supported on the lines in L . There will be two cases to consider, as to how the popular set of lines L , alias ratios from $A : A$, is defined. This is why throughout the paper we use the notation l for the individual members of the ratio set.

Ratio set case. In order to establish the estimates in Theorem 1 involving the ratio set the notation L will stand for the set of lines through the origin, supporting at least $\frac{1}{2}|A|^2|A : A|^{-1}$ points of $A \times A$ each. The subset P of $A \times A$ supported on these

lines is then such that $|P| \geq \frac{1}{2}|A|^2$, plus one has $|A|/2 \leq |L| \leq |A : A|$. Let us also use the notation N for the maximum number of points per line, clearly $N \leq |A|$. It follows from (5) that if $n(l)$ denotes the number of elements of P supported on a line $l \in L$, that is the number of realisations of the ratio $l = \frac{y}{x}$, with $x, y \in A$, then

$$(6) \quad \sum_{l \in L} n^2(l) \gg \frac{|A|^4}{|A : A|}.$$

Product set case. In order to establish the estimates in Theorem 1 involving the product set, the same notations P, L, N will apply to slightly differently defined quantities, as follows. The set P will be a “popular multiplicative energy” subset of $A \times A$, constructed by the standard dyadic pigeonholing procedure by popularity, in terms of supporting points of $A \times A$, of all lines through the origin. The elements of P are those points in $A \times A$ which are supported on the set L of lines through the origin, supporting between $N/2$ and N points each, with $|L|$ and N being such that

$$(7) \quad |L|N^2 \gg \frac{E_*(A)}{\log |A|} \geq \frac{|A|^4}{|A \cdot A| \log |A|}.$$

The exponent in the difference-set estimates (4) versus the product set is slightly worse than the one versus the ratio set. The technical reason is that the difference-product set estimate will require an additional step, expressed in the following lemma, regarding the quantities L, N in the product set case.

Lemma 2. *There exists L, N satisfying (7) and such that*

$$(8) \quad \frac{1}{2} \frac{|A|^2}{|A \cdot A|} \leq N \ll \frac{|A - A|^2 |A \cdot A|}{|A|^3}.$$

A variant of Lemma 2 can be found in the recent works [9], [12] and represents the well known approach the sum-product problem, due to Elekes ([1]). This approach, described in the Appendix to this paper, would itself only result in the exponent $\frac{5}{4}$. We relegate the proof of Lemma 2, as well as some discussion of the Elekes approach to the Appendix to this paper.

2.1. Proof of the estimates (3). Without loss of generality, as we are not pursuing optimal constants in the estimates, we may assume that the set $A \subset \mathbb{C}$ is located in a reasonably small angular sector, of angular width ϵ , around the real axis, with the vertex at 0. The quantity ϵ should not be viewed as $o(1)$: it will by far suffice to have, say $\epsilon = \frac{1}{10}$.

Let L be the above defined set of popular lines through the origin, identified with their slopes, i.e. members of the ratio set, whether we are in the ratio (cf. (6)) or the product set case (cf. (7)). So L is a point set in \mathbb{C} . Our proof is based on the following two key observations.

Claim. *Firstly, if l_1, l_2 are two distinct members of $L \subset \mathbb{C} \cong \mathbb{R}^2$, with some realisations $l_1 = \frac{y_1}{x_1}$ and $l_2 = \frac{y_2}{x_2}$, then the point $z = \frac{y_1 + y_2}{x_1 + x_2}$ lies in \mathbb{C} in some ϵ -neighbourhood of the straight line segment $[l_1, l_2]$ in \mathbb{R}^2 , to be described shortly. Then, secondly, if we consider a tree T of minimal Euclidean length, spanning the*

point set $L \subset \mathbb{R}^2$, then for ϵ small enough, the above neighbourhoods of its $|L| - 1$ edges do not intersect.

We will give a more precise statement of Claim below.

This yields indeed a bona fide generalisation of the construction of Solymosi in [15], where all the ratios being real, the set L would lie on a line in \mathbb{R}^2 , the x -axis, the edges of its minimal spanning tree being the consecutive segments between the vertices. Thus, owing to the basic fact that if one also has alternative representations of the two ratios as $l_1 = \frac{y'_1}{x'_1}$ and $l_2 = \frac{y'_2}{x'_2}$, then $\frac{y'_1+y'_2}{x'_1+x'_2} \neq \frac{y_1+y_2}{x_1+x_2}$, if $[l_1, l_2]$ denotes an edge of the tree T , assuming the claim yields the following inequality:

$$(9) \quad |A + A|^2 \geq \sum_{[l_1, l_2] \in T} n(l_1)n(l_2).$$

The tree T has $|L| - 1$ edges, and hence in the ratio set case, where $n(l_1) \geq \frac{|A|^2}{2|A:A|}$, one gets, for some $l_0 \in L$

$$(10) \quad |A + A|^2 \geq \frac{|A|^2}{2|A:A|} \sum_{l_2 \in L \setminus \{l_0\}} n(l_2) \gg \frac{|A|^4}{|A:A|},$$

since also $n(l_0) \leq |A|$. This proves the first inequality in (3), modulo the claim.

In the product set case, the claim would imply, by (7) and (9), that

$$(11) \quad |A + A|^2 \gg \frac{E_*(A)}{\log |A|} \geq \frac{|A|^4}{|A \cdot A| \log |A|},$$

thus proving the second inequality in (3).

It remains to prove the above Claim by making its observations precise.

Suppose $x_1, x_2, y_1, y_2 \in A$, $\frac{y_1}{x_1} = l_1 \in L$, $\frac{y_2}{x_2} = l_2 \in L$. Then, with $u = x_2/x_1$, we have

$$(12) \quad \frac{y_1 + y_2}{x_1 + x_2} = \frac{y_1 + y_2}{x_1(1 + u)} = \frac{l_1}{1 + u} + l_2 \frac{u}{1 + u} = l_1 + (l_2 - l_1) \frac{u}{1 + u}.$$

Since we've assumed that $|\arg x_1|, |\arg x_2| < \frac{\epsilon}{2}$, clearly u lies in the angular wedge $W_\epsilon = \{z : |\arg z| < \epsilon\}$ and therefore $\frac{u}{1+u}$ lies in the image of W_ϵ , further denoted as M_ϵ , under the Möbius map $z' = \frac{z}{1+z}$.

A straightforward calculation shows that M_ϵ is a meniscus around the line segment $[0, 1]$, formed by the intersection of two open discs centered respectively at $z_\pm = (\frac{1}{2}, \pm \frac{\epsilon}{2\epsilon})$, with equal radii $|z_\pm|$.

Hence, by (12),

$$\frac{y_1 + y_2}{x_1 + x_2} \in l_1 + (l_2 - l_1)M_\epsilon,$$

which geometrically means that the point $z = \frac{y_1+y_2}{x_1+x_2}$ lies inside the meniscus-shaped region formed around the straight line segment $[l_1, l_2]$ in \mathbb{R}^2 by two open discs, each passing through the segment's endpoints l_1, l_2 , and whose centres are positioned on the bisector to the segment $[l_1, l_2]$, at the distance $\frac{|l_2-l_1|}{2\epsilon}$ from its midpoint, on both

sides. Let us now T be a minimum length tree spanning the set $L \subset \mathbb{R}^2$. That is, T has $|L| - 1$ edges, which are straight line segments between pairs of distinct vertices in the set L ; there are no loops in T , and for any pair of distinct vertices $l_1, l_2 \in L$, there is a unique path connecting them. Plus, the sum of the Euclidean lengths of the edges attains its minimum, over all trees spanning L .

For the rest of the geometric argument, let us use, the uppercase Latin letters A, B, C, D, \dots for the vertices of T , without reference to the rest of the paper, where A, B, \dots are sets.

We observe that T does not contain intersecting edges. Indeed, assume that $[AB]$ and $[CD]$ are edges of T and $(AB) \cap (CD) \neq \emptyset$. We choose a shortest path connecting an endpoint of $[AB]$ with an endpoint of $[CD]$. Without loss of generality, it is a path connecting B and C . Then the shortest path connecting A and D contains the edges $[AB]$ and $[CD]$. These edges can be deleted and replaced by the edges $[AC]$ and $[BD]$ with a shorter sum without violating connectivity or creating loops. This contradicts the minimality of T .

Let $e = [l_1, l_2]$ be an edge of T , and $M_e = l_1 + (l_2 - l_1)M_\epsilon$ denote the corresponding meniscus around it. We shall further show that for a pair of distinct edges, the menisci around them cannot intersect, provided that ϵ is small enough.

For each edge e of the tree T , let us replace the meniscus M_e by a larger rhombus-shaped region R_e , containing M_e , where R_e is a narrow rhombus, whose major diagonal is $[l_1, l_2]$ and the minor one has length ϵ . The fact that $M_e \subset R_e$ comes from elementary Euclidean geometry. If $e = [AB]$, then $R_e = R_{[AB]}$ be the rhombus around it.

First off, the rhombi around adjacent edges cannot intersect, because of the well-known fact that in a minimum spanning tree the angle between adjacent edges is at least $\frac{\pi}{3}$. Indeed, if the intersection took place at the vertex A for the edges $[AB]$ and $[AC]$, then one of the angles in the triangle ABC would be bigger than $\frac{\pi}{3}$ and the edges opposite to it in T could be deleted and replaced by the shorter edge $[BC]$, without violating connectivity or creating loops. This contradicts the minimality of T .

So, suppose there is a pair $[AB]$ and $[CD]$ of non-adjacent edges, but such that $R_{[AB]} \cap R_{[CD]} \neq \emptyset$.

One easy observation is the following lemma.

Lemma 3. *The vertices C, D cannot lie inside a circle with the diameter $[AB]$.*

Proof. Indeed, suppose, say C lies inside the circle with the diameter $[AB]$. Then the angle ACB is obtuse. Hence, the edge $[AB]$ can be deleted and replaced in the tree T by one of the shorter line segments $[AC]$ or $[BC]$. More precisely, if the unique path from A to C in T incorporates $[AB]$, then $[AB]$ should be replaced by $[AC]$, and by $[BC]$ otherwise. This contradicts the minimality of T . \square

Without loss of generality assume that the length $|AB| = 1$, as well as $|AB| \geq |CD|$.

Double the width of $R_{[AB]}$: let $R'_{[AB]}$ be the rhombus whose major diagonal is still $[AB]$ and whose width is twice the width of the rhombus $R_{[AB]}$. So $R_{[AB]} \subset R'_{[AB]}$.

Let E, F be the two vertices of $R'_{[AB]}$, other than A, B . For $\epsilon < \frac{1}{4}$, the points E, F lie inside the circle with the diameter $[AB]$. Then, by Lemma 3, the vertices C, D must both lie outside the rhombus $R'_{[AB]}$, and hence the angle α between the edges $[AB]$ and $[CD]$ is such that $\tan \alpha < 2\epsilon$. In other words, if ϵ is small, the two edges $[AB]$ and $[CD]$ are almost parallel.

Hence, by Lemma 3 and the assumption that $|CD| \leq |AB|$, a very rough estimate as to where the vertices C, D can be located tells one that one of them, say C , must lie within the circle of radius 3ϵ around A , and the other, that is D , within the circle of radius 3ϵ around B . But this is a contradiction, because if $\epsilon < \frac{1}{3}$, the edge $[AB]$, whose length is one, in T can be deleted and replaced in T by a shorter $[AC]$ or by $[BD]$. Once again, the replacement will be $[AC]$ if the unique path from A to C in T incorporates $[AB]$; it will be $[BD]$ otherwise. This once again contradicts the minimality of T .

We have exhausted all the possibilities for the mutual alignment of a pair of edges $[AB]$ and $[CD]$, each one has led to a contradiction with the minimality of the tree T . This proves the claim, namely that for two distinct edges $e_1, e_2 \in T$, the menisci surrounding them do not intersect: $M_{e_1} \cap M_{e_2} = \emptyset$.

Therefore, we have shown that for any $e \in T$, with $e = [l_1, l_2]$, and any realisations $l_1 = \frac{y_1}{x_1}$ and $l_2 = \frac{y_2}{x_2}$, the ratio $\frac{y_1 + y_2}{x_1 + x_2}$ cannot be realised in terms of any other edge of the tree T .

Inequalities (9)–(11) now complete the proof of the estimates (3). \square

2.2. Proof of the estimates (4).

2.2.1. Lemmata. The main tool to prove the estimates (4) is the Szemerédi-Trotter incidence theorem. For any set \mathcal{P} of points and any set of \mathcal{L} straight lines in a plane let

$$I(\mathcal{P}, \mathcal{L}) = \{(p, l) \in \mathcal{P} \times \mathcal{L} : p \in l\}$$

be the set of incidences.

Theorem 4 (Szemerédi and Trotter [17]). *The maximum number of incidences in \mathbb{R}^2 is bounded as follows:*

$$(13) \quad |I(\mathcal{P}, \mathcal{L})| \ll (|\mathcal{P}||\mathcal{L}|)^{\frac{2}{3}} + |\mathcal{P}| + |\mathcal{L}|.$$

In particular, if \mathcal{P}_t (or \mathcal{L}_t) denote the sets of points (or lines) incident to at least $t \geq 1$ lines (or points) of \mathcal{L} (or \mathcal{P}), then

$$(14) \quad \begin{aligned} |\mathcal{P}_t| &\ll \frac{|\mathcal{L}|^2}{t^3} + \frac{|\mathcal{L}|}{t}, \\ |\mathcal{L}_t| &\ll \frac{|\mathcal{P}|^2}{t^3} + \frac{|\mathcal{P}|}{t}. \end{aligned}$$

Let us note that the linear in $|\mathcal{P}|, |\mathcal{L}|$ terms in the estimates (13, 14) are essentially trivial and usually of no interest in the sense of being dominated by the non-linear ones, whenever these estimates are being used. This is also the case in this paper.

The Szemerédi-Trotter theorem is also true in full generality in the plane over \mathbb{C} . This was proved by Tóth ([19]). A more modern proof came out in a recent

paper by Zahl ([20]). In the particular case, where the point set is a Cartesian product, Solymosi ([14], Lemma 1) observed that the proof of the \mathbb{C}^2 version of the Szemerédi-Trotter theorem is considerably more straightforward than dealing with an arbitrary point set in \mathbb{C}^2 . Although the geometric part of the forthcoming proof closely follows the construction in ([14]), the point sets to which we apply the theorem are not necessarily Cartesian products, so strictly speaking we are using here the general version of the Szemerédi-Trotter theorem in \mathbb{C}^2 of Tóth and Zahl. The estimates (14) will be further used in \mathbb{C}^2 without additional comments.

One can easily develop a weighted version of the estimates of the Szemerédi-Trotter theorem, see Iosevich et al. ([5]). Suppose each line $l \in \mathcal{L}$ has been assigned a weight $m(l) \geq 1$. The number of weighted incidences $i_m(\mathcal{P}, \mathcal{L})$ is obtained by summing over the set $I(\mathcal{P}, \mathcal{L})$, with each pair (p, l) being counted $m(l)$ times. Suppose, the total weight of all lines is W and the maximum weight per line is $\mu > 0$. Then it's easy to conclude that the worst possible case for the weighted incidence estimate is the one, when there are $\lceil \frac{W}{\mu} \rceil$ lines of equal weight μ , hence the following theorem (see [5] for details).

Theorem 5. *The maximum number of weighted incidences between a point set \mathcal{P} and a set of lines \mathcal{L} , with the total weight W and maximum weight per line μ is*

$$(15) \quad i_m(\mathcal{P}, \mathcal{L}) \ll \mu^{\frac{1}{3}}(|\mathcal{P}|W)^{\frac{2}{3}} + \mu|\mathcal{P}| + W.$$

The other main ingredient to establish the estimates (4) comes from a purely additive-combinatorial observation by Shkredov and Schoen ([10], Lemma 3.1), which has recently allowed for a series of state-of-the art improvements in incremental progress towards a number of open questions in field combinatorics in [10], [13], [11].

This observation is the content of the following Lemma 6, quoting which requires some notation also used in the sequel. Throughout the rest of this section A, B denote any sets in an Abelian group, the group operation being the addition. In the context of the field \mathbb{C} , Lemma 6 will apply as to the addition operation, so the following notations E will stand for the additive energy, rather than the multiplicative energy E_* , which has already been used in the proof of the sum-set estimate in (3).

For any $d \in A - A$, set

$$(16) \quad A_d = \{a \in A : a + d \in A\}.$$

Denote

$$E(A, B) = |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 - a_2 = b_1 - b_2\}|,$$

referred to as the additive energy of A, B . By the Cauchy-Schwarz inequality, rearranging the terms in the above definition of $E(A, B)$, one has

$$(17) \quad E(A, B)|A - B| \geq |A|^2|B|^2.$$

Indeed, if d is an element of $A - B$ and $n(d)$ its the number of realisations as a difference of a pair of elements from $A \times B$, (17) follows from the fact that

$$(18) \quad E(A, B) = \sum_{d \in A - B} n^2(d).$$

Also, $E(A, A) = E(A)$ is referred to as the (additive) energy of A . In this case note that according to the notation (16), $n(d) = |A_d|$.

Also useful will be the “cubic energy” of A , which is

$$(19) \quad E_3(A) = |\{(a_1, \dots, a_6) \in A \times \dots \times A : a_1 - a_2 = a_3 - a_4 = a_5 - a_6\}|.$$

This definition implies that ([11], Lemma 2)

$$(20) \quad E_3(A) = \sum_{d \in A-A} E(A, A_d).$$

To see this, fix any $d = a_1 - a_2$ in (19) and observe that if one is to count every representation $d = a_3 - a_4$ as many times as $a_3, a_4 \in A_d$ for some d , this will happen exactly $n(d)$ times, for different $d = a_5 - a_3 = a_6 - a_4$ in (19).

The following statement is part of the Corollary 3 in [11].

Lemma 6. *One has the following inequality, for any $D' \subset A - A$:*

$$(21) \quad \sum_{d \in D'} |A_d| |A - A_d| \geq \frac{|A|^2 \left(\sum_{d \in D'} |A_d|^{\frac{3}{2}} \right)^2}{E_3(A)}.$$

Proof. To verify (21) observe that by the Cauchy-Schwarz inequality (17) applied to the sets A, A_d for each d :

$$\sqrt{|A - A_d|} \sqrt{E(A, A_d)} \geq |A| |A_d|.$$

Multiplying both sides by $\sqrt{|A_d|}$ and summing over $d \in D'$, then applying once again the Cauchy-Schwarz inequality to the left-hand side yields

$$\sqrt{\sum_{d \in D'} |A_d| |A - A_d|} \sqrt{\sum_{d \in D'} E(A, A_d)} \geq |A| \sum_{d \in D'} |A_d|^{\frac{3}{2}}.$$

squaring both sides and using (20) does the job. \square

Remark 7. Inequality (21) suggests that the cubic energy estimate from above can be quite useful, since $A - A_d \subseteq (A - A) \cap (A - A - d)$. The latter observation, which [11] credits to Katz and Koester ([7]) means that the left-hand side in (21) provides a lower bound for $E(A, A - A)$. Indeed, each $d \in A - A$ has $|A_d|$ representations $d = u - v$ as an element of $A - A$, with $v \in A_d$. The same d also has at least $|A - A_d|$ representations as an element of $(A - A) - (A - A)$. Indeed, given d , for any $v \in A_d$ and $a \in A$ one can write $d = (u - a) - (v - a)$, with $|A - A_d|$ distinct values for the second bracket. Hence, if $n(d)$ is the number of representations of d as an element of $A - A$ and $n'(d)$ – as an element of $(A - A) - (A - A)$, then

$$(22) \quad E(A, A - A) = \sum_d n(d) n'(d) \geq \sum_d |A_d| |A - A_d|.$$

In the context of our paper, we will also produce an upper bound, using the Szemerédi-Trotter theorem.

From now on, D' in (21) will stand for a popular subset of the difference set $A - A$, namely

$$(23) \quad D' = \left\{ d \in A - A : |A_d| \geq \frac{1}{2} \frac{|A|^2}{|A - A|} \right\}.$$

Then

$$\sum_{d \in D'} |A_d|^{\frac{3}{2}} \gg \left(\frac{|A|^2}{|A - A|} \right)^{\frac{1}{2}} \sum_{d \in D'} |A_d| \gg \left(\frac{|A|^2}{|A - A|} \right)^{\frac{1}{2}} |A|^2.$$

Hence, by (22)

$$(24) \quad E(A, A - A) \gg \frac{|A|^8}{|A - A| E_3(A)}.$$

Remark 8 (Some applications of Lemma 6). The estimate (24) enabled Schoen and Shkredov [11] to achieve progress on the sum set of a convex set problem. If $A = f([1, \dots, N])$, where f is a strictly convex real-valued function, then $|A - A| \gg |A|^{\frac{5}{3} - o(1)}$. The conjectured exponent in the convex set sum set problem is $2 - o(1)$. Li [8] – see also his recent work with Roche-Newton [9] – pointed out that the approach of [11] can be adapted to the sum-product problem, using a variant of the well-known sum-product construction by Elekes [1], which is briefly presented in the Appendix to this paper, since it is used here to obtain the last one of the estimates (4). This improved the exponent $\frac{5}{4}$, obtained by Elekes within his construction to $\frac{14}{11} - o(1)$. The same exponent $\frac{14}{11} - o(1)$ had been coincidentally obtained in Solymosi's work [15], as stated in (2) above. Also recently Jones and Roche-Newton [6] applied the estimate (24) to improve the best known lower bound on the size of $A(A + 1)$.

2.2.2. The main body of the proof of (4). In either of the ratio or product set cases above consider the sum set of the correspondingly defined point set $P \subset \mathbb{C}^2$ (containing the points of $A \times A$ supported on the set of popular lines L , each supporting at most N points) with some other set Q , with $|Q| \geq |P|$. (In the sequel $Q = -P$ or $P - P$). To obtain the vector sums, one translates the lines from L to each point of Q , getting thereby some set \mathcal{L} of lines with $|\mathcal{L}| \leq |L||Q|$.

In both the ratio and product set cases, it can be assumed that $|L| \gg N$. It is clear in the ratio set case, where $N \leq |A| \ll |L|$. In the product set case $|P| \approx |L|N$, since each popular line contains at least $N/2$ and at most N points. Assume that $N \gg |L|$. Since N satisfies (8) and LN^2 is bounded from below by (7), it follows that

$$|A - A|^6 |A \cdot A|^4 \gg \frac{|A|^{13}}{\log |A|},$$

which is far better than (4).

The Szemerédi-Trotter theorem, namely (13), enables one to estimate $|\mathcal{L}|$ from below. We have the following estimate for the number of incidences

$$(25) \quad |L||Q| \leq |I(Q, \mathcal{L})| \ll |\mathcal{L}|^{\frac{2}{3}} |Q|^{\frac{2}{3}} + |\mathcal{L}| + |Q|.$$

Since it can be assumed that $|L|$ is bigger than an absolute constant (as the target estimates (4) are up to absolute constants), the term $|Q|$ in (25) cannot dominate the estimate. Nor can the term $|\mathcal{L}|$, for otherwise $|\mathcal{L}| > |Q|^2$. This, since by construction of \mathcal{L} one has $|\mathcal{L}| \leq |L||Q|$, would imply $|L| > |Q|$, but in our set-up $|Q| \geq |P| \geq |L|$.

Thus it follows from (25) that

$$(26) \quad |\mathcal{L}| \gg |L|^{\frac{3}{2}} |Q|^{\frac{1}{2}}.$$

Let us call the number of points of Q on a particular line $l \in \mathcal{L}$, the weight $m(l)$ of l . The total weight W of all lines in the collection \mathcal{L} is by construction equal to $|L||Q|$.

The lines in \mathcal{L} have been given weights, because the same line $l \in \mathcal{L}$ can contribute to the same vector sum in $P + Q$ at most $\max(N, m(l))$ times. Hence, let us lower the weights of lines, which are “too heavy”: whenever $m(l) \geq N$, redefine it as N . Therefore, W denoting the total weight of the lines in \mathcal{L} , and \bar{m} the average weight per line, one has

$$(27) \quad W \leq |L||Q|, \quad \bar{m} = \frac{W}{|\mathcal{L}|} \ll \sqrt{\frac{|Q|}{|L|}},$$

where the second estimate has used (26).

The Szemerédi-Trotter theorem, namely (14), tells one that the weight distribution over \mathcal{L} obeys the inverse cube law. I.e., for $t \leq N$, one has

$$(28) \quad |\mathcal{L}_t| = |\{l \in \mathcal{L} : m(l) \geq t\}| \ll \frac{|Q|^2}{t^3} + \frac{|Q|}{t} \ll \frac{|Q|^2}{t^3},$$

as since $N \ll \sqrt{|Q|}$, the trivial term $\frac{|Q|}{t}$ gets dominated by the first term. It also follows from (28), via the standard dyadic summation in t , that the total weight $W(\mathcal{L}_t)$ supported on the lines from \mathcal{L}_t is bounded by

$$(29) \quad W(\mathcal{L}_t) \ll \frac{|Q|^2}{t^2},$$

Now, let us look at the set $\mathcal{P}(\mathcal{L})$ of all pair-wise intersections of lines from \mathcal{L} , and for an intersection point $p \in \mathcal{P}(\mathcal{L})$ of some $k \geq 2$ lines l_1, \dots, l_k look at the sum of the weights of the lines that intersect there:

$$m(p) = \sum_{i=1}^k m(l_i).$$

For any point set $\mathcal{P} \subseteq \mathcal{P}(\mathcal{L})$, the number of weighted incidences $i_m(\mathcal{P}, \mathcal{L})$ between \mathcal{P} and \mathcal{L} is the sum over all pairs $(p, l) \in I(\mathcal{P}, \mathcal{L})$, counting each pair (p, l) with the weight $m(l)$.

The inverse cube weight distribution over the set of lines \mathcal{L} enables one to use the Szemerédi-Trotter theorem rather efficiently for counting weighted incidences, similar to how it was done in the paper of Iosevich et al. (See [5], Lemma 6).

Lemma 9. *Suppose, $|Q| \geq |P|$, and the weights of lines in \mathcal{L} have been capped by N . For $x \in P + Q$, let $n(x)$ be the number of realisations of x as a sum. Then for $t : N \ll t \leq |P|$,*

$$(30) \quad |\{x \in P + Q : n(x) \geq t\}| \ll \frac{|L|^{\frac{3}{2}}|Q|^{\frac{5}{2}}}{t^3},$$

Proof. The condition $|Q| \geq |P| \gg N^2$ (which holds in both the ratio and product set cases) ensures that (28) is valid, since for all $l \in \mathcal{L}$, $m(l) \leq N$. Observe that for any point set \mathcal{P} , the number of weighted incidences $i_m(\mathcal{P}, \mathcal{L})$ of \mathcal{L} with \mathcal{P} can be bounded from above using dyadic decomposition of \mathcal{L} by weight in excess of \bar{m} , via

$$(31) \quad i_m(\mathcal{P}, \mathcal{L}) \ll \sum_{j=0}^{\log_2 N - \log_2 \bar{m}} i_m(\mathcal{P}, \mathcal{L}_{2^j \bar{m}}),$$

where $\mathcal{L}_{\bar{m}}$ is the subset of \mathcal{L} containing all those lines whose weight does not exceed \bar{m} , while for $j \geq 1$, $\mathcal{L}_{2^j \bar{m}}$ denotes the dyadic group of lines whose weights are approximately $2^j \bar{m}$. The notation i_m refers to weighted incidences, and to estimate each individual term $i_m(\mathcal{P}, \mathcal{L}_{2^j \bar{m}})$ one can use Theorem 5. The quantity $2^j \bar{m}$ then replaces the maximum weight μ in (15). The total weight W in (15), in view of (29), will be replaced by the quantity

$$(32) \quad W_{2^j \bar{m}} \ll \frac{|Q|^2}{2^{2j} \bar{m}^2},$$

the weight supported on the dyadic group $\mathcal{L}_{2^j \bar{m}}$.

Thus

$$(33) \quad i_m(\mathcal{P}, \mathcal{L}_{2^j \bar{m}}) \ll (2^j \bar{m})^{\frac{1}{3}} (|\mathcal{P}| W_{2^j \bar{m}})^{\frac{2}{3}} + 2^j \bar{m} |\mathcal{P}| + W_{2^j \bar{m}}.$$

Using (32), it follows that in the summation (31), the term $j = 0$ dominates the net contribution of the first and the third terms in the estimate (33). Conversely, the dominant value of the second term in (33) corresponds to the maximum value N of the lines' weight. Thus

$$(34) \quad \begin{aligned} i_m(\mathcal{P}, \mathcal{L}) &\ll \bar{m}^{\frac{1}{3}} (|\mathcal{P}| W)^{\frac{2}{3}} + N |\mathcal{P}| + W \\ &\ll |\mathcal{P}|^{\frac{2}{3}} |L|^{\frac{3}{6}} |Q|^{\frac{5}{6}} + N |\mathcal{P}| + |L| |Q|, \end{aligned}$$

using (27).

Recall now that for $x \in P + Q \subseteq \mathcal{P}(\mathcal{L})$, with $n(x)$ denoting the number of realisations of a sum set element x , one always has $n(x) \leq m(x)$, where $m(x)$ is the weight of x as a member of $\mathcal{P}(\mathcal{L})$. I.e.

$$(35) \quad \{x \in P + Q : n(x) \geq t\} \subseteq (\mathcal{P}_t \equiv \{p \in \mathcal{P}(\mathcal{L}) : m(p) \geq t\}).$$

Applying the incidence bound (34) to the point set \mathcal{P}_t with the lower bound $t |\mathcal{P}_t|$ for $i_m(\mathcal{P}_t, \mathcal{L})$, one sees that for $t \gg N$ the term $N |\mathcal{P}|$ in the right-hand side of (34) cannot possibly dominate the remaining terms. Hence for $t \gg N$ one has

$$(36) \quad |\mathcal{P}_t| \ll \frac{|L|^{\frac{3}{2}}|Q|^{\frac{5}{2}}}{t^3} + \frac{|L||Q|}{t}.$$

This, in view of (35), establishes the claim (30) of the Lemma, as long as the first term in the estimate (36) dominates, which is clearly the case for $t \leq \sqrt[4]{|L||Q|^3}$. But for larger t , one has to be slightly more careful with the trivial term $\frac{|L||Q|}{t}$, which can be, in fact, refined for $t \gg |L|\bar{m}$ (which is less than the threshold $t \sim \sqrt[4]{|L||Q|^3}$ in (36)).

Indeed, the lines in \mathcal{L} come in $|L|$ possible directions, and therefore no more than $|L|$ lines can be incident to a single point in $\mathcal{P}(\mathcal{L})$. Hence, lines from a dyadic group $\mathcal{L}_{2^j\bar{m}}$ can contribute only a small proportion to the total number of weighted incidences supported on the sets \mathcal{P}_t , with $t \gg |L| \cdot (2^j\bar{m})$.

This means that for $t \sim |L| \cdot (2^i\bar{m})$, $i \geq 1$ one needs only a “tail” estimate for the right-hand side of (31) as to the point set \mathcal{P}_t , the sum beginning with $j = i$. Thus, for such large t , the estimate (34) for the number of weighted incidences can be modified as follows: its first two terms are kept as they are, while the latter term gets replaced by $W_{\frac{t}{|L|}}$, the total weight of supported on lines, whose weight is at least $\frac{t}{|L|}$. The latter, by (29) is bounded as $O(\frac{|Q|^2|L|^2}{t^2})$. Hence, for such t the estimate (34) becomes

$$(37) \quad i_m(\mathcal{P}_t, \mathcal{L}) \ll |\mathcal{P}_t|^{\frac{2}{3}} |L|^{\frac{3}{6}} |Q|^{\frac{5}{6}} + N|\mathcal{P}_t| + \frac{|Q|^2|L|^2}{t^2}.$$

It follows from the lower bound $t|\mathcal{P}_t| \leq i_m(\mathcal{P}_t, \mathcal{L})$ that for $t \gg N$,

$$(38) \quad |\mathcal{P}_t| \ll \frac{|L|^{\frac{3}{2}}|Q|^{\frac{5}{2}}}{t^3} + \frac{|Q|^2|L|^2}{t^3}.$$

Since $|Q| \geq |P| \geq |L|$, the first term in the last estimate (38) dominates the second term. This, in view of (35), proves (30). \square

Let us now apply the estimate (30) with $Q = -P$; of these two the case $Q = -P$ will serve to estimate $E_3(P)$.

The assumptions of Lemma 9 are satisfied, and therefore

$$(39) \quad E_3(P) = \sum_{x \in P-P} n^3(x) \ll N^2|P|^2 + |L|^{\frac{3}{2}}|P|^{\frac{5}{2}} \sum_{j=1}^{2\log_2|A|} \frac{2^{3j}}{2^{3j}} \ll |L|^{\frac{3}{2}}|P|^{\frac{5}{2}} \log |A|,$$

since in both the ratio and product set cases $N^2 \ll \sqrt{|P||L|^3}$.

Using this, the first formula in (24), with A replaced by P , yields:

$$(40) \quad E(P, P-P) \gg \frac{|P|^{\frac{11}{2}}}{|L|^{\frac{3}{2}}|P-P| \log |A|}.$$

On the other hand, one can use Lemma 9 with $Q = P - P$ and estimate the quantity $E(P, P - P)$ from above. Then for any $t \gg N$:

$$(41) \quad E(P, P-P) \ll |P||P-P|t + \frac{|L|^{\frac{3}{2}}|P-P|^{\frac{5}{2}}}{t},$$

and choosing

$$t = \frac{|P - P|^{\frac{3}{4}} |L|^{\frac{3}{4}}}{\sqrt{|P|}} \gg N,$$

to match the two terms in (41) yields

$$(42) \quad E(P, P - P) \ll \sqrt{|P|} |P - P|^{\frac{7}{4}} |L|^{\frac{3}{4}}.$$

Combining this with (40) results in

$$(43) \quad |P - P|^{\frac{11}{4}} |L|^{\frac{9}{4}} \gg \frac{|P|^5}{\log |A|}.$$

To obtain the first estimate of (4) it suffices to note that one has (with what the notations P, L stand for in the ratio set case) $|P| \gg |A|^2$, $|L| \leq |A : A|$, plus $|P - P| \leq |A - A|^2$.

In the product set case, where $|P| \approx |L|N$, the estimate (43) becomes

$$(44) \quad |A - A|^{\frac{11}{2}} \gg \frac{(|L|N^2)^{\frac{11}{4}}}{\sqrt{N} \log |A|}.$$

Lemma 2 now supplies a non-trivial upper bound on N . Substituting the bounds from (7) and (8) into (44) then yields the last estimate of (4) and completes the proof of Theorem 1. \square

3. APPENDIX. PROOF OF LEMMA 2

The lower bound of (8) follows from the Cauchy-Schwarz inequality application (5). All the lines through the origin, whose slopes are the ratios $l \in A : A$, whose number of realisations $n(l) < \frac{1}{2} \frac{|A|^2}{|A \cdot A|}$, can contribute no more than

$$\frac{1}{2} \frac{|A|^2}{|A \cdot A|} \sum_l n(l) \leq \frac{1}{2} \frac{|A|^4}{|A \cdot A|}$$

to the multiplicative energy $E_*(A)$, that is, by (5), no more than one half thereof. Hence, the multiplicative energy of A supported on those lines through the origin, which correspond to ratios $l \in A : A$, whose number of realisations $n(l) \geq \frac{1}{2} \frac{|A|^2}{|A \cdot A|}$, is at least $\frac{1}{2} \frac{|A|^4}{|A \cdot A|}$.

On the other hand, it was proven in [9], [12] that the multiplicative energy coming from the lines through the origin, supporting at least N points of $A \times A$ (that from the set of all ratios $l \in A : A$ such that $n(l) \geq N$), is $O\left(\frac{|A - A|^2 |A|}{N}\right)$. This settles the upper bound in (8) and proves Lemma 2.

For completeness sake, a simple version of the proof of the upper bound in (8) for N is given below. This bound was derived in [9], [12] via the Szemerééd-Trotter type estimates for convex functions, using a particular example of a convex function, the exponential. Let us show that the bound in question, in fact, represents a variant of the well known construction of Elekes [1] apropos of sum-products, which gave the

exponent $\frac{5}{4}$ implicit in the bounds (8). The notation in the forthcoming argument is somewhat independent from the rest of the paper.

Consider a set A , not containing zero and a set of lines $\mathcal{L} = \{y = \frac{d+x}{a}\}$, where d is an element of the difference set $A - A$ (or the sum set $A + A$, the modification required being trivial) and $a \in A$. Clearly there are $|A - A||A|$ lines. Therefore, the number of points in a set \mathcal{P}_t , where at least $t \leq |A|$ lines from \mathcal{L} intersect is, by (14), bounded as

$$(45) \quad |\mathcal{P}_t| \ll \frac{|A - A|^2 |A|^2}{t^3}.$$

Suppose now that L_N is the subset of all ratios from $A : A$, each $l \in L_N$ having $n(l) \in [\frac{N}{2}, N]$ realisations. So for each $l \in L_N$, one has $l = \frac{a'_i}{a_i}$, where the index i runs over $n(l)$ distinct values. Given $l \in L_N$, for every $a \in A$, one has $l = \frac{(a'_i - a) + a}{a_i}$, for $i = 1, \dots, n(l)$. I.e., the point in the plane with coordinates (a, l) is incident to at least $n(l)$ lines from \mathcal{L} , these lines being identified by the pairs $(d_i = a'_i - a, a_i)$, with $i = 1, \dots, n(l)$.

Hence $A \times L_N \subseteq \mathcal{P}_{\frac{N}{2}}$, and it follows from (45) that

$$(46) \quad |L_N| \ll \frac{|A - A|^2 |A|}{N^3}.$$

Thus the multiplicative energy supported on L_N is $O\left(\frac{|A - A|^2 |A|}{N}\right)$. Comparing this with the lower bound in terms of $\frac{|A|^4}{|A \cdot A|}$ gives the upper bound for N in (8). \square

Remark 10. This approach by Elekes is symmetric with respect to the two field operations in \mathbb{C} , which can be swapped in the estimate (8), by considering the set of lines as $\mathcal{L} = \{y = lx - a\}$, where $(l, a) \in (A : A) \times A$.

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